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ON THE QUASI-INVARIANT PHENOMENA IN THE AXISYMMETRICAL INTERFACE CRACK PROBLEM AND ITS APPLICATION TO FIXED END CYLINDER INVESTIGATION

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Abstract—A penny-shaped crack in the central part of a semi-infinite cylinder with a fixed end is under consideration. The model of an interface crack with a contact ring near its tip is used. Similar to the plane problem, the quasi-invariant of stress intensity factor combinations relative to the contact ring width is obtained, and the energy release rate is given. By using Fourier and Hankel integral transforms, the system of singular integral equations is obtained. This system is solved numerically for different loads and material properties. The value of the quasi-invariant at the crack tip and stress intensity factors at the corner edge of the cylinder are found.

1. INTRODUCTION

Much attention has recently been focused on the interface crack problem which is of great importance in nonhomogeneous and composite materials investigation. Using classical models of crack leads to the oscillatory singularity [see e.g. Williams (1959); Cherepanov (1962); Erdogan (1963), (1965); Sih and Rice (1964); Rice and Sih (1965); England (1965); Rice (1988)] when the crack tip is approached. The "contact zone" model [see e.g. Comninou (1977), (1978); Dundurs and Comninou (1979)] makes it possible to avoid the singularity of oscillatory type but, nevertheless, the length of contact zone for a "mode 1" crack is extremely small which leads to difficulties in numerical analysis. A simple way of using the "contact zone" model for interface crack numerical analysis was proposed by Loboda (1993), where the quasi-invariant with respect to contact zone length was found.

The goal of this work is to extend the results of Loboda (1993), obtained for the interface crack plane problem, to the investigation of a penny-shaped crack settled in the interface of two mediums. In the first part of the paper, the quasi-invariant is obtained for the case of the axisymmetrical problem by the investigation of a simple problem for half-spaces. Next this quasi-invariant is applied to the investigation of a semi-infinite cylinder with a penny-shaped crack in the center of its fixed end.

2. A PENNY-SHAPED CRACK AT THE INTERFACE OF DISSIMILAR HALF-SPACES

Consider the axisymmetric problem of two dissimilar half-spaces, $z > 0$ and $z < 0$, containing an interfacial penny-shaped crack of radius b. The crack is loaded with uniform pressure of intensity $P(r)$. Let E and ν be the Young's modulus and Poisson's ratio of the upper half-space and lower half-space which is absolutely rigid. By introducing the zone of frictionless contact of crack surfaces $a < r < b$ near the crack tip, the boundary conditions at the plane $z = 0$ can be written as

$$\sigma_{zz}(r,0) = -P(r), \quad \sigma_{rz}(r,0) = 0, \quad r \leqslant a \tag{1}$$

$$u_z(r,0) = 0, \quad u_r(r,0) = 0, \quad b \leqslant r \leqslant h \tag{2}$$

$$u_z(r,0) = 0, \quad \sigma_{rz}(r,0) = 0, \quad a < r < b, r > h. \tag{3}$$

The zones of frictionless contact for $r > h$ are used for the convenience of numerical analysis; this assumption is not a principal one, and according to St. Venant's principle they will not influence the state of stress near the crack, provided $h \gg b$.

In order to obtain the solution of the problem, the following unknown functions are introduced:

$$q_1(r) = \sigma_{rz}(r,0), \quad q_2(r) = \frac{\partial u_z}{\partial r}(r,0). \tag{4}$$

Using the representations of the components of the stresses and displacements in terms of the stress function $\Phi(r, z)$, and applying the Hankel transform, with respect to the variable r, to the equation

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{\partial^2}{\partial z^2}\right)^2 \Phi(r,z) = 0, \tag{5}$$

we obtain an ordinary differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}z^2}-p^2\right)^2 \bar{\Phi}(p,z) = 0, \tag{6}$$

where

$$\bar{\Phi}(p,z) = \int_0^\infty rI_0(pr)\Phi(r,z)\,\mathrm{d}r.$$

The general solution of eqn (6) can be chosen in the form

$$\bar{\Phi}(p,z) = [A(p)+B(p)z]\,\mathrm{e}^{-pz}.$$

The system of equations for the determination of $A(p)$ and $B(p)$ follows from eqn (4) and, after Hankel transform application, can be written as

$$v\frac{\mathrm{d}^2\bar{\Phi}}{\mathrm{d}z^2}+(1-v)p^2\bar{\Phi} = \frac{1}{p}\bar{q}_1(p)$$

$$(1-2v)\frac{\mathrm{d}^2\bar{\Phi}}{\mathrm{d}z^2}-2(1-v)p^2\bar{\Phi} = \frac{-E}{p(1+v)}\bar{q}_2(p), \tag{7}$$

where

$$\bar{q}_j(p) = \int_{c_j}^{d_j} yI_1(py)q_j(y)\,\mathrm{d}y, \quad j = 1,2$$

$$c_1 = b, \quad d_1 = h, \quad c_2 = 0, \quad d_2 = a.$$

We took into account that $q_1(r) = 0$ for $r < b \cup r > h$ and $q_2(r) = 0$ for $r > a$.

By substituting $A(p)$ and $B(p)$ we obtain the function $\bar{\Phi}(p, z)$, and by using the inverse transform we can write the expressions for $\Phi(r, z)$ in terms of the unknown functions $q_1(r)$ and $q_2(r)$. Then the expressions for normal stress σ_{zz} and radial displacement u_r at $z = 0$ take the following forms:

$$2u_r(r,0) = -\sum_{j=1}^{2} \xi_{1j}\eta_{1j}\int_{c_j}^{d_j} yq_j(y)\,dy\int_0^\infty I_1(py)I_1(pr)\,dp$$

$$2\sigma_{zz}(r,0) = -\sum_{j=1}^{2} \xi_{2j}\eta_{2j}\int_{c_j}^{d_j} yq_j(y)\,dy\int_0^\infty pI_1(py)I_0(pr)\,dp, \tag{8}$$

where

$$\xi_{11} = 3-4\nu,\quad \xi_{12} = -(1-2\nu),\quad \xi_{21} = -\xi_{12},\quad \xi_{22} = 1$$

$$\eta_{11} = \frac{1}{2\mu(1-\nu)},\quad \eta_{12} = \frac{1}{1-\nu},\quad \eta_{21} = -\eta_{12},\quad \eta_{22} = -\frac{E}{1-\nu^2}.$$

The inner integrals in eqn (8) can be written in closed form in terms of complete elliptic integrals $K(x)$ and $E(x)$ (the first and second kind, respectively). Using the asymptotic behavior of $K(x)$ and $E(x)$ as $x \to 1$ allows us to extract Cauchy and logarithmic singularities.

Next we introduce the function $\tilde{u} \equiv \partial u_r/\partial r + u_r/r$ and satisfy condition (1) and $\tilde{u}(r,0) = 0$. The system of singular integral equations can be written in the form

$$\sum_{j=1}^{2} \xi_{ij}\eta_{ij}\int_{c_j}^{d_j}\left\{\frac{1}{r-y} + \frac{1}{r+y}\ln|r-y| + M_1(r,y)\right\}q_j(y)\,dy = F_i(r), \tag{9}$$

where

$$F_1(r) = 0,\quad F_2(r) = -2\pi P(r), r\in[c_i,d_i], i = 1,2$$

$$M_1(r,y) = \frac{1}{r-y}E\left(\frac{2\sqrt{ry}}{r+y}\right) - \frac{1}{r+y}K\left(\frac{2\sqrt{ry}}{r+y}\right) - \frac{1}{r-y} - \frac{1}{r+y}\ln|r-y|,$$

μ is the shear modulus, M_1 $(r,$ $y)\in H$, where H is a class of function which satisfies the Holder condition.

The system (9) must be adjoined by the consistency conditions

$$\sum_{j=1}^{2} \xi_{1j}\eta_{1j}\int_{c_j}^{d_j}\left\{-\frac{b^2+y^2}{b(b+y)}\ln|b-y| + M_2(b,y)\right\}q_j(y)\,dy = 0, \tag{10}$$

where

$$M_2(b,y) = \frac{b^2+y^2}{b(b+y)}K\left(\frac{2\sqrt{by}}{b+y}\right) - \frac{y+b}{b}E\left(\frac{2\sqrt{by}}{b+y}\right) + \frac{b^2+y^2}{b(b+y)}\ln|r-y|$$

$$M_2(b,y)\in H.$$

The unknown functions $q_j(y)$ are assumed to have integrable singularity at points c_j and d_j and can be expressed as

$$q_j(y) = \frac{q_j^*(y)}{\sqrt{(y-c_j)(d_j-y)}},\quad q_j^*(y)\in H,\quad (j = 1,2). \tag{11}$$

Stress intensity factors K_1 and K_2 are defined as

Table 1

λ	$\tilde{K}_2$	$\tilde{K}_1$	$\tilde{K}$
10^{-1}	−0.1144	0.6160	0.6532
10^{-2}	−0.2517	0.5729	0.6489
10^{-3}	−0.3578	0.5134	0.6445
10^{-4}	−0.4155	0.4540	0.6330

$$K_1 = \lim_{r\to a+0} \sqrt{2(r-a)}\,\sigma_{zz}(r,0), \quad K_2 = \lim_{r\to b-0} \sqrt{2(r-b)}\,\sigma_{rz}(r,0) \tag{12}$$

and

$$K_1^b = \lim_{r\to b-0} \sqrt{2(b-r)}\,\sigma_{zz}(r,0). \tag{13}$$

Taking into account that the normal and shear stresses are defined in terms of the unknown functions $q_i(y)$, and employing eqn (11), the stress intensity factors can be rewritten in the form

$$K_1 = \eta_{22}q_2^*(a)/\sqrt{2a}, \quad K_2 = q_1^*(b)\sqrt{2/(h-b)}$$

$$K_1^b = -\eta_{21}q_1^*(b)/\sqrt{2(h-b)}. \tag{14}$$

Using the method described in Loboda and Sheveleva (1991), the system of singular integral equations (9)–(10) is reduced to the system of linear algebraic equations which was solved numerically.

The analysis of the numerical solutions shows that the main conclusions in this case coincide with the results which have been received in the plane case. Specifically, it is noticed that, similar to the plane problem (Loboda, 1993), the value of

$$K = \sqrt{\omega K_1^2 + K_2^2}, \tag{15}$$

where $\omega = [4(1-\nu)^2]/(3-4\nu)$ for $\lambda \in [\lambda_*, \lambda_0]$, $\lambda_* \approx 10^{-2}$ is nearly invariable. Here $\lambda = (b-a)/b$, and λ_0 is the value of parameter λ for which the normal stress in the contact zone is compressive and there is no overlapping of crack faces.

The quasi-invariantness of parameter K was confirmed in the axisymmetrical case by the numerical calculation of K_1, K_2 and K for various λ and ν. Here some results for $b/h = 0.25$, $P(r) \equiv P_0$ = constant are shown. The dimensionless values of stress intensity factors $\tilde{K}_i = K_i/(P_0\sqrt{b})$ and quasi-invariant $\tilde{K} = K/(P_0\sqrt{b})$, which have been obtained for $\nu = 0.3$ and $\nu = 0.0283$, are given in Tables 1 and 2. In the last column of Table 2 values of N are shown (the order of the system of linear algebraic equations) which were necessary for the determination of K_i with an accuracy of 1%. The quasi-invariantness of K and equation $K_1 = 0$ (for $\lambda = \lambda_0$) give the possibility to determine the main parameter of fracture $K_{20} = K_2|_{\lambda=\lambda_0}$ as $K_{20} \approx K$. On the other hand, according to Loboda (1993), we have the following formulae for the energy release rate

Table 2

λ	$\tilde{K}_2$	$\tilde{K}_1$	$\tilde{K}$	N
10^{-1}	−0.2211	0.5748	0.6937	6
10^{-2}	−0.4509	0.4435	0.6787	14
10^{-3}	−0.5941	0.2689	0.6690	40
$\lambda_0 = 1.05\times10^{-4}$	−0.6653	0	0.6660	80

$$G = \frac{\pi(3-4\nu)}{16\mu(1-\nu)}K^2 \quad \text{and} \quad G_0 = \frac{\pi(3-4\nu)}{16\mu(1-\nu)}K_{20}^2, \tag{16}$$

which allows us to apply an energy criteria of fracture for interface problems. Thus quasi-invariantness of K can be used for the numerical solution of the practical problems.

An example of this parameter application is given in the next section.

3. THE CRACK ALONG THE FIXED END OF AN ELASTIC CIRCULAR CYLINDER

A semi-infinite right circular cylinder of radius R is partially bonded to an absolutely rigid half-space. The penny-shaped crack of radius b is situated in the central part of the lower end and the remaining contact area is perfectly bonded to the half-space. The lateral surface of the cylinder $r = R$, is subjected to an arbitrary axisymmetric load

$$\sigma_{rz}(R, z) = P_1(z), \quad \sigma_{rr}(R, z) = P_2(z). \tag{17}$$

Let the frictionless contact of their surfaces take place in the neighbourhood of the crack tip $a < r < b$. In such a way the boundary conditions at the plane $z = 0$ are as follows:

$$\begin{aligned} &\sigma_{zz}(r,0) = 0 \quad \sigma_{rz}(r,0) = 0, \quad r \leqslant a \\ &u_z(r,0) = 0, \quad u_r(r,0) = 0, \quad b \leqslant r \leqslant R \\ &u_z(r,0) = 0, \quad \sigma_{rz}(r,0) = 0, \quad a < r < b. \end{aligned} \tag{18}$$

Introducing the same unknown functions as above and using the technique based upon Fourier and Hankel transforms, the normal stress σ_{zz} and the function $\tilde{u}$ at the plane $z = 0$ can be found as follows:

$$\begin{aligned} 2\pi\tilde{u}(r,0) &= \eta_{11}\int_b^R q_1(y)K_{11}(r,y)\,\mathrm{d}y + \eta_{12}\int_0^a q_2(y)K_{12}(r,y)\,\mathrm{d}y + \frac{2}{\mu}f_1(r) \\ 2\pi\sigma_{zz}(r,0) &= \eta_{21}\int_b^R q_1(y)K_{21}(r,y)\,\mathrm{d}y + \eta_{22}\int_0^a q_2(y)K_{22}(r,y)\,\mathrm{d}y - 4f_2(r), \end{aligned} \tag{19}$$

where

$$\begin{aligned} K_{ij}(r,y) &= \frac{1}{r-y} + \frac{1}{r+y}\ln|r-y| + M_1(r,y) + 2yL_{ij}(r,y) \\ L_{ij}(r,y) &= \int_0^\infty k_{ij}(r,y,p)/D(p)\,\mathrm{d}p \\ k_{ij}(r,y,p) &= H_{1j}(y,p)\,l_{i1}(r,p) + H_{2j}(y,p)l_{i2}(r,p) \\ f_i(r) &= \int_0^\infty [l_{i1}(r,p)\bar{P}_2(p) - l_{i2}(r,p)\bar{P}_1(p)]/D(p)\,\mathrm{d}p \\ \bar{P}_1(p) &= \int_0^\infty P_1(z)\sin pz\,\mathrm{d}z, \; \bar{P}_2(p) = \int_0^\infty P_2(z)\cos pz\,\mathrm{d}z. \end{aligned} \tag{20}$$

The expressions for the functions $H_{ij}(y, p)$, $l_{ij}(r, p)$, $D(p)$ are given in the Appendix.

The kernels $L_{ij}(r, y)$ contain singularities as $y \to R$ and $r \to R$. These singularities can be extracted by using the asymptotic behaviour of the integrand $L_{ij}(r, y)$ as $p \to \infty$.

The singular part of kernels $L_{ij}(r, y)$, denoted by $L_{ij}^\infty(r, y)$, are obtained by considering the asymptotic expressions of

$$k_{ij}^{\infty}(r,y,p) = k_{ij}(r,y,p)/D(p)|_{p\to\infty}$$

$$D^{\infty}(p) = \mathrm{e}^{2pR}/(2\pi)$$

$$k_{11}^{\infty}(r,y,p) = \frac{\mathrm{e}^{-p(2R-r-y)}}{2\sqrt{ry}}\left\{2p^2(R-y)(R-r)+p[(1-4v)(R-y)+(3-4v)(R-r)]\right.$$
$$\left.+(8v^2-8v+2)+\frac{\lambda_{11}}{2pR}\right\}$$

$$k_{12}^{\infty}(r,y,p) = -\frac{\mathrm{e}^{-p(2R-r-y)}}{2\sqrt{ry}}\left\{2p^2(R-y)(R-r)+p[(1-4v)(R-y)-(R-r)]\right.$$
$$\left.+2v+\frac{\lambda_{12}}{2pR}\right\}$$

$$k_{21}^{\infty}(r,y,p) = -\frac{\mathrm{e}^{-p(2R-r-y)}}{2\sqrt{ry}}\left\{2p^2(R-y)(R-r)+p[(3-4v)(R-r)-3(R-y)]\right.$$
$$\left.+(-4+6v)+\frac{\lambda_{21}}{2pR}\right\}$$

$$k_{22}^{\infty}(r,y,p) = -\frac{\mathrm{e}^{-p(2R-r-y)}}{2\sqrt{ry}}\left\{2p^2(R-y)(R-r)-p[3(R-y)+(R-r)]+2+\frac{\lambda_{22}}{2pR}\right\}, \tag{21}$$

where

$$\lambda_{11} = 12v^2-16v+8,\quad \lambda_{12} = 8v^2-5v-1,\quad \lambda_{21} = 8v^2-11v+5,\quad \lambda_{22} = 8v-6.$$

Using eqns (21) and performing the integration in eqns (20), the corresponding singular part of the kernels becomes

$$L_{11}^{\infty}(r,y) = \frac{1}{2\sqrt{ry}}\left\{\frac{8v^2-12v+3}{2R-r-y}+\frac{6(R-r)}{(2R-r-y)^2}-\frac{4(R-r)^2}{(2R-r-y)^3}-\frac{\lambda_{11}}{2R}\ln(2R-r-y)\right\}$$

$$L_{12}^{\infty}(r,y) = \frac{1}{2\sqrt{ry}}\left\{\frac{1-2v}{2R-r-y}+\frac{2(1+2v)(R-r)}{(2R-r-y)^2}-\frac{4(R-r)^2}{(2R-r-y)^3}-\frac{\lambda_{12}}{2R}\ln(2R-r-y)\right\}$$

$$L_{21}^{\infty}(r,y) = \frac{1}{2\sqrt{ry}}\left\{\frac{-7+6v}{2R-r-y}+\frac{(10-4v)(R-r)}{(2R-r-y)^2}-\frac{4(R-r)^2}{(2R-r-y)^3}-\frac{\lambda_{21}}{2R}\ln(2R-r-y)\right\}$$

$$L_{22}^{\infty}(r,y) = \frac{1}{2\sqrt{ry}}\left\{\frac{-1}{2R-r-y}+\frac{6(R-r)}{(2R-r-y)^2}-\frac{4(R-r)^2}{(2R-r-y)^3}-\frac{\lambda_{22}}{2R}\ln(2R-r-y)\right\}. \tag{22}$$

Therefore, the singular integral equations can be written in the form

$$\sum_{j=1}^{2}\eta_{ij}\int_{c_j}^{d_j}\left\{\frac{\xi_{ij}}{r-y}+\frac{\xi_{ij}}{r+y}\ln|r-y|-2yL_{ij}^{\infty}(r,y)+K_{ij}(r,y)\right\}q_j(y)\,\mathrm{d}y = B_i(r),$$
$$i = 1,2, r\in[c_i,d_i], \tag{23}$$

where

$$B_1(r) = -\frac{2}{\mu} f_1(r), \quad B_2(r) = 4f_2(r).$$

The system of integral equations given by eqn (23) must be adjoined by the consistency conditions $u_r(b, 0) = 0$ and $q_2(0) = 0$. By using the expression for radial displacement and extracting the dominant part of the kernels, we obtain

$$\sum_{j=1}^{2} \eta_{1j} \int_{c_j}^{d_j} \left\{ -\frac{\xi_{1j}(b^2+y^2)}{b(b+y)} \ln|b-y| + K_{3j}(b,y) \right\} q_j(y)\,\mathrm{d}y = B_0, \quad q_2(0) = 0, \tag{24}$$

where

$$K_{3j}(b,y) = \xi_{1j} \left\{ \frac{b^2+y^2}{b(b+y)} K\left(\frac{2\sqrt{by}}{b+y}\right) - \frac{y+b}{b} E\left(\frac{2\sqrt{by}}{b+y}\right) + \frac{b^2+y^2}{b(b+y)} \ln|r-y| \right\} + 2yL_{3j}(b,y)$$

$$L_{3j}(r,y) = \int_0^\infty k_{3j}(r,y,p)/D(p)$$

$$k_{3j}(r,y,p) = H_{1j}(y,p) l_3(r,p) + H_{2j}(y,p) l_4(r,p)$$

$$B_0 = \frac{2}{\mu} \int_0^\infty [l_3(b,p)\bar{P}_2(p) - l_4(b,p)\bar{P}_1(p)]/D(p)\,\mathrm{d}p$$

$$l_3(r,p) = Rp\,\{RI_1(pr)I_0(pR) - rI_0(pr)I_1(pR)\} + 2(1-\nu)RI_1(pr)I_1(pR)$$

$$l_4(r,p) = Rp\,\{rI_0(pr)I_0(pR) - RI_1(pr)I_1(pR)\} - rI_0(pr)I_1(pR) - (1-2\nu)RI_1(pr)I_0(pR).$$

Next we assume that $q_j(y)$ has integrable singularities at the points c_j, d_j and they can be expressed as

$$q_j(y) = q_j^*(y)/W_j(y),$$

where

$$q_j^*(y) \in H,\ W_1(y) = (R-y)^\alpha (y-b)^\beta,\ W_2(y) = \sqrt{y}(a-y)^\gamma.$$

Following the procedure outlined in Muskhelishvili (1953), we obtain the following transcendental equation:

$$(3-4\nu)\cos\pi\alpha - (8\nu^2 - 12\nu + 3) + 8\alpha + 2\alpha^2 = 0$$

giving the power of singularity α and $\beta = \gamma = 0.5$.

New unknown functions $\tilde{q}_i(y)$ are introduced for numerical solutions (23)–(24)

$$\tilde{q}_i(y) = q_i(y) - \mu_{i1}\theta_{i1}(y) - \mu_{i2}\theta_{i2}(y), \tag{25}$$

where

$$\mu_{11} = \frac{q_1^*(b)}{(R-b)^{\alpha+\beta}}, \quad \mu_{12} = \frac{q_1^*(R)}{(R-b)^{\alpha+\beta}}, \quad \theta_{11}(y) = \left(\frac{R-y}{y-b}\right)^\beta, \quad \theta_{12}(y) = \left(\frac{y-b}{R-y}\right)^\alpha$$

$$\mu_{21} = \frac{q_2^*(0)}{a}, \quad \mu_{22} = \frac{q_2^*(a)}{a}, \quad \theta_{21} = \sqrt{\frac{a-y}{y}}, \quad \theta_{22}(y) = \sqrt{\frac{y}{a-y}}.$$

Table 3

b/R	$\tilde{K}_1$	$\tilde{K}_2$	$\tilde{K}$	$\tilde{K}_1^h$	$\tilde{K}_2^R$	$\tilde{K}_1^R$
0.3	1.006	−0.5448	1.182	−0.1557	−0.2992	0.9725
0.5	1.054	−0.6409	1.273	−0.1831	−0.3112	1.011
0.7	1.203	−0.7894	1.483	−0.2256	−0.3576	1.162

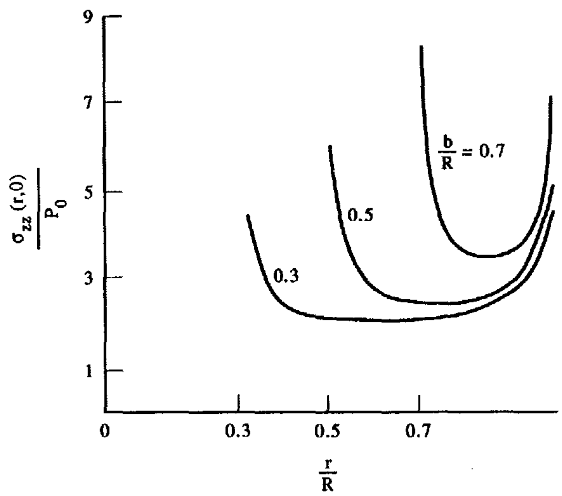

Fig. 1. Normal stress along the fixed end of the cylinder for various sizes of the contact zone.

By substituting eqn (25) into eqns (23) and (24), we reduce eqns (23) and (24) to the system of singular integral equations in terms of $\tilde{q}_i(y)$ and parameters μ_{ij}.

Stress intensity factors K_1, K_2 and quasi-invariant K are defined by eqns (12), (13) and (15). In addition, we introduce stress intensity factors K_1^R and K_2^R as follows:

$$K_1^R = \lim_{r\to R-0}[2(R-r)]^{\alpha}\,\sigma_{zz}(r,0)$$

$$K_2^R = \lim_{r\to R-0}[2(R-r)]^{\alpha}\,\sigma_{rz}(r,0). \tag{26}$$

In terms of parameters μ_{ij} we can write

$$K_2^R = [2(R-b)]^{\alpha}\mu_{12}$$

$$K_1^R = \frac{\eta_{21}K_2^R}{2\sin(\pi\alpha)}[(1-2\nu)\cos(\pi\alpha)+7-6\nu-4\alpha(2-\nu)+2\alpha^2].$$

Numerical results have been received for various values of ν, b and R. The dimensionless values of stress intensity factors $\tilde{K}_i = K_i/(P_1^0\sqrt{b})$, $\tilde{K} = K/(P_1^0\sqrt{b})$, $\tilde{K}_1^h = K_1^h/(P_1^0\sqrt{b})$ and $\tilde{K}_i^R = K_i^R/(P_1^0\sqrt{b})$ are shown in Table 3. They have been calculated for $\lambda = 10^{-2}$, $\nu = 0.3$, $P_1(z) = P_1^0\delta(z-d)$, $P_2(z) = 0$, $d = 20$, $P_1^0 =$ constant. Figure 1 shows the variation of the normal stress σ_{zz} $(r, 0)$ along the bonded zone for various values of b/R. It must be pointed out that the normal stress has singularities near the crack tip and the circumference of the lower end of the cylinder, $r = R$, is the same as the function $q_1(r)$ and $q_2(r)$.

Using the value of $\tilde{K}$ we can find the value of $K = \tilde{K}P_1^0\sqrt{b}$ and, due to the quasi-invariantness of K, the values of $K_{20} \approx K$ and G_0.

REFERENCES

Cherepanov, G. P. (1962). The stress state in a heterogeneous plate with slits (in Russian). *Isvestia AN USSR, OTN, Mechan. Mashin.* **1,** 131–137.

Comninou, M. (1977). The interface crack. *J. Appl. Mech.* **44,** 631–636.

Comninou, M. (1978). The interface crack in shear field. *J. Appl. Mech.* **45,** 287–290.

Dundurs, J. and Comninou, M. (1979). Revision and perspective of interface problem investigation (in Russian). *Mechanica Comp. Materialov* **3,** 387–396.

England, A. H. (1965). A crack between dissimilar media. *J. Appl. Mech.* **32,** 400–402.

Erdogan, F. (1963). Stress distributions in non-homogeneous elastic plane with cracks. *J. Appl. Mech.* **30,** 232–236.

Erdogan, F. (1965). Stress distribution in bonded dissimilar materials with cracks. *J. Appl. Mech.* **32,** 403–410.

Loboda, V. V. (1993). The quasi-invariant in the theory of interface cracks. *Engng Fract. Mech.* **44,** 573–580.

Loboda, V. V. and Sheveleva, A. E. (1991). The axisymmetrical problem for a penny-shaped crack with contact surfaces near its tip (in Russian). *Phys. Chem. Mech. Mater.* **3,** 55–61.

Muskhelishvili, N.I. (1953). *Singular Integral Equations.* Noordhoff, Groningen.

Rice, J. R. (1988). Elastic fracture mechanic concepts for interfacial cracks. *J. Appl. Mech.* **55,** 98–103.

Rice, J. R. and Sih, G. C. (1965). Plane problems of cracks in dissimilar media. *J. Appl. Mech.* **32,** 418–423.

Sih, G. C. and Rice, J. R. (1964). The bending of plates of dissimilar materials with cracks. *J. Appl. Mech.* **31,** 477–482.

Williams, M. L. (1959). The stresses around a fault or cracks in dissimilar media. *Bull. Seismol. Soc. Am.* **49,** 199–204.

APPENDIX

Given below are expressions which are needed in Section 3.

$$H_{11}(y,p) = \frac{1}{R}\{[R^2p^2+4(1-v)]I_1(py)K_1(pR)+(3-2v)RpI_1(py)K_0(pR)-yRp^2I_0(py)K_0(pR)-ypI_0(py)K_1(pR)\}$$

$$H_{12}(y,p) = -\frac{1}{R}\{[R^2p^2+2(1-v)]I_1(py)K_1(pR)+RpI_1(py)K_0(pR)-yRp^2I_0(py)K_0(pR)-ypI_0(py)K_1(pR)\}$$

$$H_{21}(y,p) = -p\{2(1-v)I_1(py)K_1(pR)+RpI_1(py)K_0(pR)-ypI_0(py)K_1(pR)\}$$

$$H_{22}(y,p) = -p\{-RpI_1(py)K_0(pR)+ypI_0(py)K_1(pR)\}$$

$$D(p) = R^2p^2I_0^2(pR)-[R^2p^2+2(1-v)]I_1^2(pR)$$

$$l_{11}(r,p) = Rp\{RpI_0(pr)I_0(pR)-rpI_1(pr)I_1(pR)-2I_0(pr)I_1(pR)\}$$

$$l_{12}(r,p) = Rp\{3I_0(pr)I_0(pR)-RpI_0(pr)I_1(pR)+rpI_1(pr)I_0(pR)\}-2(2-v)I_0(pr)I_1(pR)-rpI_1(pr)I_1(pR)$$

$$l_{21}(r,p) = Rp\{RpI_0(pr)I_0(pR)-rpI_1(pr)I_1(pR)-2vI_0(pr)I_1(pR)\}$$

$$l_{22}(r,p) = Rp\{(1+2v)I_0(pr)I_0(pR)-RpI_0(pr)I_1(pR)+rpI_1(pr)I_0(pR)\}-rpI_1(pr)I_1(pR)-2I_0(pr)I_1(pR).$$